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THE REMARKABLE INCIRCLE OF A TRIANGLE

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1. Introduction

The *incircle* of a triangle is that circle which just touches all three sides of the triangle. Figure 1 shows the incircle for a triangle. It is easy to see that the center of the incircle (*incenter*) is at the point where the angle bisectors of the triangle meet. In this note we refer to a right triangle in which all three sides are relatively prime as a *Pythagorean triangle*. The two most familiar are the 3, 4, 5 and the 5, 12, 13 triangles. In this paper we discuss several unusual facts regarding triangles all of which are related to properties of the incircle.

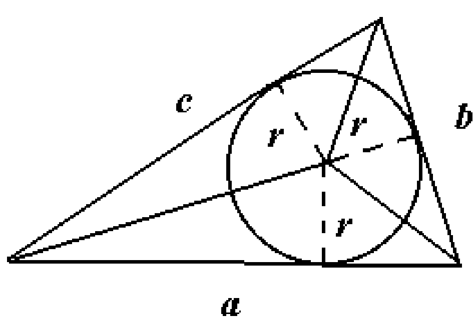

Figure 1: The incircle

A known, but not well advertised theorem is that the radius (*inradius*) of the incircle of a Pythagorean triangle is an integer. After rediscovering this feature of

Pythagorean triangles, we found it tucked away in a few number theory texts [1,3]. For example, the inradius of the 3, 4, 5 triangle is one and the inradius of the 5, 12, 13 triangle is two. One purpose of this note to give two surprisingly simple proofs that the inradius of a Pythagorean triangle is an integer. These appear in section 2.

When discussing Pythagorean triangles, we usually concentrate our attention on the sides of the triangle. But what can be said concerning the angles of a Pythagorean triangle? Surprisingly, the study of the incircle leads to a fundamental result about these angles, and this is discussed in section 3.

Heron's formula for the area of a triangle in terms of its sides a, b and c and the semi-perimeter $s = (a+b+c)/2$ is $A = \sqrt{s(s-a)(s-b)(s-c)}$. In section 4 we show that the incircle gives us a simple geometric meaning for the factors that occur under this radical.

Finally, in section 5, we show how to find non-right triangles with integer sides and integer inradius. This is achieved by starting with any two Pythagorean triangles, and then creating a new triangle from magnifications and unions of the original two.

We assume that the reader is familiar with the fact that the sides of Pythagorean triangles can be found from the following simple algorithm [2, 3]. Let m and n be natural numbers, with $m > n$. Then the three sides a, b and c of a Pythagorean triangle are given by the equations

$$a = 2mn,$$

$$b = m^2 - n^2, \text{ and}$$

$$c = m^2 + n^2.$$

For example, if we take $n = 1$ and $m = 2$, we get the 3, 4, 5 Pythagorean triangle. If we let

$n = 2$ and $m = 3$ we get the 5, 12, 13 triangle.

This paper explores several relatively unknown yet interesting theorems which are not usually included in the high school curriculum. The proofs of these theorems integrate the use of geometry, algebra and trigonometry. The classroom teacher should find the interweaving of these topics an excellent example of their interdependence. Some of the ideas presented in the paper should serve as a review for students and many of the ideas will probably be new. This article is appropriate for high school students in an advanced geometry or precalculus course.

2. Derivation of the inradius

We begin by finding the radius of the incircle in any triangle.

Theorem 1:

Let a, b, and c be the sides of a triangle. Let A *be the area of the triangle and* $s = (a+b+c)/2$ *be the semi-perimeter. Then the radius r of the incircle is given by*

(2.1) $$r = \frac{A}{s} \text{ , and}$$

(2.2) $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} .$$

Proof:

Refer to Figure 1. The area A of the triangle is the sum of the areas of the three interior triangles made by the angle bisectors. This sum is

$$A = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{a+b+c}{2} r = sr$$

Solving this last relation for r we get (2.1). Recalling Heron's formula (see [4] and [5]) for the area of any triangle in terms of the semi-perimeter s :

$$A = \sqrt{s(s-a)(s-b)(s-c)} \,,$$

we see that (2.2) follows from (2.1) immediately. This completes the proof.

Next, we determine the inradius of a Pythagorean triangle. We give two nice, simple derivations. The first can be found in [3], and the second was shown to us by an undergraduate student, David Thibault.

Theorem 2:

Let m and n be positive integers with m > n. Let $a = 2mn$, $b = m^2 - n^2$ *and* $c = m^2 + n^2$ *be the sides of a Pythagorean triangle. Then the radius of the incircle r is given by the integer*

(2.3) $$r = n(m-n) \,.$$

Proof 1:

Refer to Figure 2. We could prove this theorem by using (2.1), but it is just as easy to work directly from the Pythagorean triangle. The area of the Pythagorean triangle is

(2.4) $$A = \frac{ab}{2} = \frac{(2mn)(m^2-n^2)}{2} = mn(m^2-n^2) \,.$$

This area is the sum of the areas of the three interior triangles made by the angle bisectors.

(2.5) $$A = \frac{(2mn)r}{2} + \frac{(m^2-n^2)r}{2} + \frac{(m^2+n^2)r}{2} \,.$$

This last result simplifies to

(2.6) $$A = m(m+n)r \,.$$

Equating (2.4) and (2.6) and solving for the inradius r we at once get

(2.7) $r = n(m-n)$, as claimed.

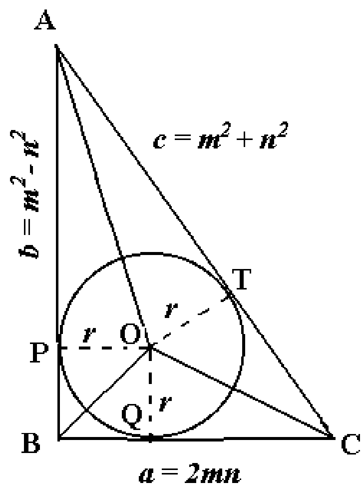

Figure 2: The three interior triangles formed from the angle bisectors.

We now give a second simple proof of Theorem 2.

Proof 2:

In Figure 2, the right triangles OQC and OTC are congruent. Also, the right triangles OPA and OTA are congruent. We see that BQ = BP = r, since BO bisects angle ABC, so QC = *a* - *r* and AP = *b*-*r*. Since AT = AP and TC = QC, we have

$$c = \text{AT} + \text{TC} = \text{AP} + \text{QC} = (b-r)+(a-r).$$

This last relation reduces to $c = a+b-2r$. Solving for *r* we get $r = \frac{a+b-c}{2}$.

Substituting the parametric relations $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$ we get $r = n(m-n)$ at once. This completes our second proof.

3. Angles of Pythagorean triangles

We now show that the half-angles of our Pythagorean triangles satisfy some interesting relations. Figure 3 shows the details of various line segments in the

Pythagorean triangle related to the incenter. The reader will have no difficulty checking the lengths of these line segments given that the inradius is $r = n(m-n)$.

From Figure 3 we see that

(3.1) $$2\alpha = \tan^{-1}\left(\frac{m^2-n^2}{2mn}\right), \text{ and}$$

(3.2) $$2\beta = \tan^{-1}\left(\frac{2mn}{m^2-n^2}\right).$$

Again, from Figure 3 we see that

(3.3) $$\alpha = \tan^{-1}\left(\frac{m-n}{m+n}\right), \text{ and}$$

(3.4) $$\beta = \tan^{-1}\left(\frac{n}{m}\right).$$

Notice that the arctangents of the angles 2α and 2β are quadratic rational functions of *m* and *n*, while the corresponding arctangents of the half-angles α and β are simpler linear rational expressions. In particular, (3.4) immediately gives us the following interesting theorem:

Theorem 3:

Let n/m be any positive rational number such that n/m < 1. *There exists a Pythagorean triangle with an acute angle such that the tangent of one half of the acute angle is equal to n/m.*

Another nice fact emerges from this analysis of the geometry of the incircle. Since $2\alpha + 2\beta = \pi/2$ we have

(3.5) $$\alpha + \beta = \frac{\pi}{4}.$$

Substituting (3.3) and (3.4) into (3.5) we get

$$(3.6) \qquad \tan^{-1}\left(\frac{m-n}{m+n}\right)+\tan^{-1}\left(\frac{n}{m}\right)=\frac{\pi}{4}.$$

This last relation has interesting consequences. When we take $m = 2$ and $n = 1$ we get the 3, 4, 5 triangle and (3.6) now reads

$$(3.7) \qquad \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}.$$

Another interesting derivation of (3.7) is found in [6].

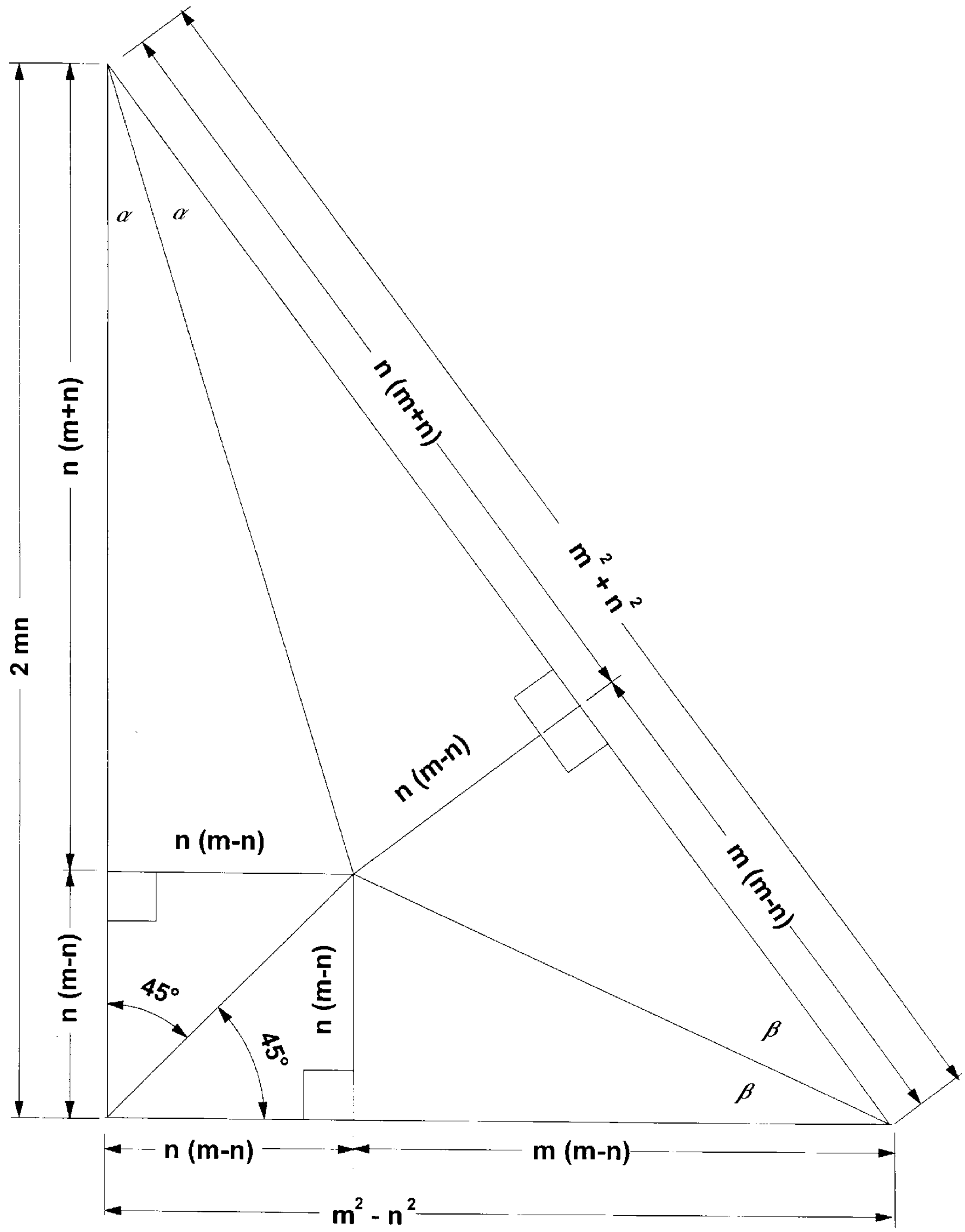

Figure 3: Details of the position of the incenter.

4. The factors in Heron's formula

We noted earlier without proof that Heron's formula for the area of the triangle shown in Figure 4 with sides *a, b* and *c* in terms of the semi-perimeter $s=(a+b+c)/2$ is

(4.1) $$A=\sqrt{s(s-a)(s-b)(s-c)}\ .$$

In this section we show the geometric meaning of the strange factors $(s-a)$, $(s-b)$ and $(s-c)$. Heron's original proof is a complex geometric argument that begins by constructing the incircle. A nice account of this is found in Dunham's book [4]. We will not derive (4.1), but will only show the significance of the factors appearing under the radical.

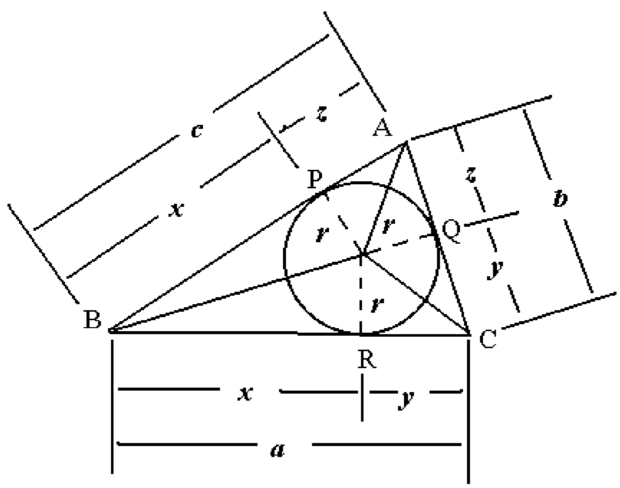

Figure 4: The incircle and Heron's formula

In Figure 4, P, Q and R are the points where the incircle touches the sides of the triangle. We will show that the factors in Heron's formula are given by:

$$s-a=z=\text{AP}=\text{AQ}$$

$$s-b=x=\text{BP}=\text{BR}$$

$$s-c=y=\text{CQ}=\text{CR}$$

From Figure 4 we see that

$$a+b+c=2s=2x+2y+2z\,,$$

and thus we have

(4.2) $$s=x+y+z\,.$$

We see at once from Figure 4 that

(4.3) $$a=x+y$$

(4,4) $$b=y+z$$

(4.5) $$c=x+z$$

Subtracting (4.3) from (4.2) we get $s-a=z$. In the same way subtracting (4.4) yields $s-b=x$, and subtracting (4.5) gives us $s-c=y$. This completes our derivation. We see that the incircle of a triangle divides the sides of the triangle into segments which are the factors appearing in Heron's formula.

5. Finding arbitrary integer triangles with integer inradius

In general, a non-Pythagorean triangle with integer sides will have an incircle whose radius is likely to be irrational. However, it is easy to create integer triangles with integer inradius as we will now show. In Figure 5 we start with two Pythagorean triangles,

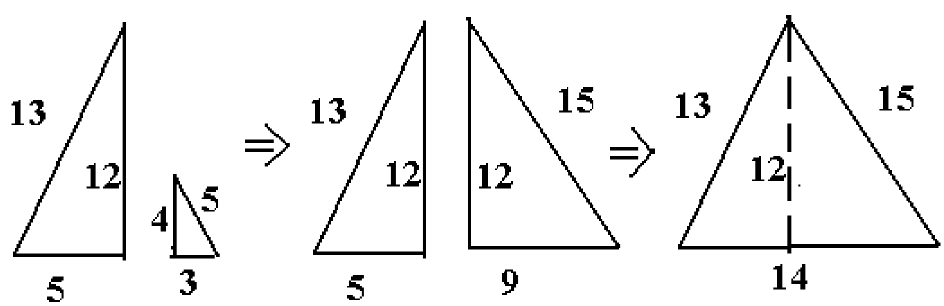

Figure 5: Constructing an integer triangle from two Pythagorean triangles

the 5, 12,13 and the 3, 4, 5. We then magnify the 3, 4, 5 by multiplying each side by 3. Now the two vertical sides are both 12, so the triangles can be joined together to form one triangle with sides 13, 14 and 15. Pythagorean triangles always have integer area, because one of the legs is always an even number. Thus the triangle we have created by joining two Pythagorean triangles together must have integer area. Referring to Figure 5 we see that this area is $A = 12 \cdot 14 / 2 = 84$. Theorem 1 tells us that the radius r of the incircle for this 13, 14, 15 triangle is given by $r = A / s$. The semi-perimeter is $s = 21$, so the inradius is $r = 84 / 21 = 4$.

We can repeat the above procedure starting with any two Pythagorean triangles. By magnifying one (or perhaps both) triangles by an integer factor, the vertical sides become equal, and so they can be joined as above. This new triangle has integer area. The inradius calculated from Theorem 1 is therefore an integer or a rational number. If r is rational, the entire triangle can be magnified by an appropriate integer factor so that the new inradius is an integer.

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